

ON THE DISCRETE TIME AND CONTINUOUS STATES MODELS*

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Abstract

We shall consider a discrete time version of Vasicek (1977) model with continuous state space. Unlike traditional approaches, we regard that continuous time and states models are approximation to the real discrete time models. This may be justified by the fact that we can only observe price of every asset in discrete time domain. In order to analyze discrete time and continuous state models, we shall modify the Ito formula so that it is suitable for the discrete time models. Using the modified Ito formula by Rao et al. (1974), we shall obtain the PDE for the continuous time approximation of the time t price of T maturity default free zero coupon bonds up to the terms as small as \sqrt{h} , where h denotes the atom of time interval.

I. Introduction

In the last 20 years, modeling a movement of the term structure of interest rate has been discussed by many authors. The papers, Black and Scholes (1973), Merton (1973), Vasicek (1977), Cox, Ingersoll and Ross (1987), and Heath, Jarrow and Morton (1992), may be regarded as the mile stones for the continuous time and states models, which we shall call the continuous models hereafter. In the continuous model, the random disturbances are assumed to follow Brownian motion. And the movement of the financial assets and the interest rates are expressed in terms of the Ito's stochastic integrals. Thus, stochastic calculus becomes a basic mathematical tool for analyzing the continuous models. Most of the continuous models seem to be adequate both for explanation and prediction of the real term structure movement as well as for pricing the derivatives. And with the market price of risks, we can transform real probability measure to the martingale probability measure. But we can only use discrete data for the empirical studies. The problem of discretization becomes a major issue of the implementation of the continuous models (cf. Kamizono and Kariya (1996)). In order to apply the results of Heath, Jarrow and Morton (1992) directly to the discrete data, Kamizono and Kariya (1996) have to restrict that the term structure of variance is of the step function. In this paper, we shall propose to discretize the model from the beginning.

The discrete time and discrete states models have been developed, for example, by Ho and Lee (1986), Hull and White (1993, 1994a, 1994b). Their versions of discrete time and states

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models have been assumed binomial or trinomial processes for the random movements of the financial assets. The models by Hull and White have been widely used both by academics and practitioners, for their models are not only easy to implement, but also rich enough to be made consistent with the initial term structure of interest rates. But in the process of adjusting the models to the given initial term structure of interest rate, their models become rather ad hoc. And it would be difficult to use their method for prediction and explanation of the real interest rate, for they have implicitly obtained martingale measure for the binomial or trinomial processes in the models, or they have explicitly assumed the risk neutral world from the very beginning.

Heath, Jarrow and Morton (1989, 1990), Amin and Morton (1994), and Amin and Bodurth (1995) also consider the discrete time and discrete state model in which the condition of Harrison and Pliska (1981) is explicitly used for the existence of no arbitrage opportunities in the market. Again it is not easy to explain the real movement of the term structure by their methods.

The other shortcoming of above discrete models are that (i) their state spaces are highly restrictive and ad hoc, so that with the assumption of complete market, it allows only a few independent assets, (ii) although, under some mild conditions, most of these discrete models can be shown to converge to its continuous counterparts as the time interval goes to zero (Heath, Jarrow and Morton (1990)), the limits depend heavily on the choice of nuisance parameters. For example, Heath, Jarrow and Morton (1990) showed in a complicated manner that the limit of Ho and Lee (1986) model exists and converges to its continuous time version if and only if their real and martingale probability are equal; $q = \pi$. And it is not clear, in general, how we can obtain a reasonable discrete model which approximates any given continuous model. This leads us to consider the problem of discrete time and continuous state approximation to the continuous model, which, in turn, will be the basis for the continuous time approximation of the discrete time models. And for this purpose, we shall start from the continuous model and consider its discrete time approximation.

Now, apart from the "philosophical" question of whether the nature is continuous or discrete, we shall ask what are the technical advantages of continuous model. Some of them would be that (i) we may be able to reduce the problems to the system of partial differential equations, so that the solutions are obtained at least numerically, and (ii) we can prove that the probability of price processes become negative in an infinitesimal time interval is negligible (Chernoff (1961)). This makes the whole analysis substantially easier than the discrete time model, where we have to consider the possibility of the price going below zero.

Except for the special case, nothing seems to be continuous.[†] Or even if the price process of any financial assets is continuous time process, we can observe its price change only at the time of tradings, which is not continuous. Since only discrete time data are observed, the models which are consistent with the observed data should be discrete in time. And the continuous time model may be regarded as an approximation to the real (discrete) one, or it is simply a limit of discrete time model as the interval of consecutive tradings goes to

[†]If the time of trading follows, for example, Poisson process independent of the price process, we can prove that the combined process may be that of continuous time process (cf. Lamperti (1977) Ch.6). This may justify the use of continuous model, but the data are not coming continuously. The details of this statement will be published elsewhere. Although they do not mention explicitly, the empirical evidence of this fact is given in Hall et al. (1977).

zero. But we may want to enjoy the advantages of the continuous models and we use them as an approximation. Now, if we approximate the discrete time process by the continuous time process, we need to evaluate the error of the approximation. As we shall see later, if we approximate a continuous model by a discrete time and continuous state model, we have an extra random term of order $O_P(\Delta t)$ as $\Delta t \rightarrow 0$, which is uncorrelated with the main random term of the order $O_P(\sqrt{\Delta t})$ as $\Delta t \rightarrow 0$, where Δt is the time between tradings. We shall start with the continuous model and consider its discrete time approximation. And we shall obtain the correction terms from the higher order Taylor series type expansion of the continuous process.

To fix the idea, we shall suppose that the time t (> 0) price of the default free zero coupon discount bond with maturity T ($> t$) is a function of the instantaneous spot interest rate $r(t)$ (cf. Vasicek (1977)). And we shall call this bond T -bond and denote its time t price $P(t, T)$. The local no arbitrage condition is achieved by constructing a locally risk free portfolio from the discount bonds with different maturities. Here the locally risk free at time t means that the time $(t + dt)$ price of the portfolio is predictable at t ; namely, infinitesimal price change between t and $t + dt$ contains no random factor. Here the key step is to eliminate the random movements from the portfolio. Using the Ito formula, it is shown that if $r(t)$ is the stochastic integral of single Brownian motion $W(t)$, the infinitesimal change of $P(t, T)$ contains one random factor $dW(t)$, which is a limit of the random term with order of magnitude $O_P(\sqrt{\Delta t})$. In order to eliminate the random part, it suffices to consider a portfolio consists of two T -bonds with different maturities. But as we shall see below, if we approximate the continuous model by discrete model (or vice versa) up to the stochastic order as small as Δt , we have two uncorrelated random factors of order $O_P(\sqrt{\Delta t})$ and $O_P(\Delta t)$ respectively.

Now let us review some basic facts about the stochastic calculus. The Ito formula analyzes the infinitesimal change of the function $Y(t) = g(t, W(t))$ between time t and $t_1 = t + \Delta t$, as Δt goes to zero, where $W(t)$ is a standard Brownian motion and $g(t, x)$ is twice continuously differentiable function. Although it must be justified mathematically via the Ito's stochastic integral, the Ito formula is basically a "Taylor series expansion" of $g(t_1, W(t_1))$ about $(t, W(t))$ up to the order as small as Δt , and then takes a L_2 limit as $\Delta t \rightarrow 0$. We have,

$$\begin{aligned}
 \Delta Y(t) &= g(t_1, W(t_1)) - g(t, W(t)) \\
 &= g_{10}(t, W_t)\Delta t + g_{01}(t, W_t)\Delta W(t) + \frac{1}{2} \{g_{20}(t, W_t)(\Delta t)^2 \\
 &\quad + 2g_{11}(t, W_t)\Delta t\Delta W(t) + g_{02}(t, W_t)(\Delta W(t))^2\} + o_P(\Delta t) \\
 &= [g_{10}(t, W_t) + g_{02}(t, W_t)]\Delta t + g_{01}(t, W_t)\Delta W(t) \\
 &\quad + \frac{1}{2}g_{02}(t, W_t) \{(\Delta W(t))^2 - \Delta t\} + o_P(\Delta t) \\
 &= [g_{10}(t, W_t) + g_{02}(t, W_t)]\Delta t + g_{01}(t, W_t)\Delta W(t) + O_P(\Delta t)
 \end{aligned} \tag{1.1}$$

as $\Delta t \rightarrow 0$, where $W_t = W(t)$, $\Delta W(t) = W(t + \Delta t) - W(t)$, and

$$g_{ij}(t, x) = \frac{\partial^{i+j}}{\partial t^i \partial x^j} g(t, x) \quad i, j = 0, 1, 2.$$

The Ito formula is then obtained by replacing the random movement $(\Delta W(t))^2$ by the deterministic quantity Δt , thus eliminating the random part of order $O_p(\Delta t)$ in the limit. This is justified by the weak law of large numbers, for $[(\Delta W(t))^2 - \Delta t]$ has mean 0 and variance $2(\Delta t)^2$. We may conclude that $[(\Delta W(t))^2 - \Delta t]$ converges to zero in probability as $\Delta t \rightarrow 0$. Replacing $\Delta W(t)$ and Δt by $dW(t)$ and dt respectively in (1-1), we arrive at the simplest version of the Ito formula.

$$dY(t) = [g_{10}(t, W_t) + g_{02}(t, W_t)] dt + g_{01}(t, W_t) dW(t) \quad (1.2)$$

Note that the random part $[(\Delta W(t))^2 - \Delta t]$ disappears only at the limit. We also note that the stochastic order of magnitude of $[(\Delta W(t))^2 - \Delta t]$ is of the order $O_p(\Delta t)$, as $\Delta t \rightarrow 0$, which is the same order of magnitude as in the predictable drift terms. Still only the former drops out from the Formula.

Suppose " h " is the smallest time span we can observe in the market and if we also suppose the discrete time model with span h , then we cannot eliminate the random part $[(\Delta W(t))^2 - \Delta t]$ from our analysis. And the Ito formula may not be the proper tool for this discrete time, continuous states model.

The above heuristic argument suggests us that in the discrete time model with the smallest time span h , the result should contain the random part $(\Delta W(t))^2 - \Delta t = O_p(h)$, if $h = \Delta t$. We shall retain this random part and then obtain the existence of No-Arbitrage condition. Since we have the second random part of magnitude as small as h , we need third bond to kill the random fluctuation of the portfolio. Thus the model requires another parameter (second market price of risks). The resulting system of partial difference equations may then be approximated by the system of partial differential equations. Here we can apply the results of the multifactor continuous models (cf. Brennan and Schwartz (1979), Takahashi (1996)).

The paper will be organized as follows. In Section 2, we shall develop some technical tools, which are simple modification of the Ito formula. We shall present our main results in Sections 3, and an easy example will be presented in Section 4.

II. The Discrete Time Ito Formula

Let (Ω, \mathcal{F}, Q) be a sufficiently rich probability space, on which an 1-dimensional standard Brownian motion $\{W(t) = W(t, \omega), t \geq 0\}$ is defined. We shall abbreviate this by $W(t) \sim BM(0, 1)$. We shall let $\{\mathcal{F}_t = \sigma(W(s), s \leq t), t \geq 0\}$ be an increasing sequence of sub σ -algebras in \mathcal{F} generated by $W(t)$. We shall also assume that $\{b(t, \cdot), t \geq 0\}$ and $\{\rho(t, \cdot), t \geq 0\}$ are \mathcal{F}_t adapted processes with

$$Q \left\{ \int_0^t |b(s, \cdot)| ds < \infty, \quad \text{for all } t \geq 0 \right\} = 1$$

and $\rho(t, \cdot) \geq 0$ a.e Q for all $t \geq 0$,

$$Q \left\{ \int_0^t \rho(s, \cdot)^2 ds < \infty, \quad \text{for all } t \geq 0 \right\} = 1.$$

Then the Ito diffusion $\{X(t) = X(t, \omega), t \geq 0\}$ may be defined by

$$dX_t = b(t, X_t)dt + \rho(t, X_t)dW_t, \quad t \geq 0, \quad X_0 = x, \quad (2.1)$$

where $X_t = X(t)$, $W_t = W(t)$ (cf. Øksendall (1992)). Now, the Ito formula states that for any twice continuously differentiable function $g(t, x)$, the process $Y_t = g(t, X_t)$ is (i) the Ito diffusion and (ii) it satisfies

$$dY_t = \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2 g_{02}(t, X_t) \right] dt + \rho(t, X_t)g_{01}(t, X_t)dW_t, \quad (2.2)$$

where, $g_{ij}(t, x) = \frac{\partial^{i+j}}{\partial t^i \partial x^j} g(t, x)$, $i, j = 0, 1, 2$. Note that heuristically speaking, the Ito formula contains the random part dW_t , which is of the order as small as \sqrt{dt} .

Now, the Ito formula is a suitable mathematical tool to analyze the local no arbitrage conditions in the continuous model. But as we have discussed in the previous section, we may need to modify the Ito formula to treat discrete time and continuous state models.

As in the previous section, let h be the smallest time span between any consecutive trades. And let us also define the set of all time points at which all economic activities are taken place,

$$\mathcal{D} = \{t_0, t_1, t_2, \dots\}, \quad t_0 = 0, \quad \Delta t = t_{i+1} - t_i = \text{integral multiple of } h \quad (2.3)$$

($i = 0, 1, 2, \dots$). In Lemma 1, we shall analyse the behavior of $\Delta Y_t = Y_{t+\Delta t} - Y_t$ on \mathcal{D} . We shall retain the terms involving $\{W_{t+\Delta t} - W_t\}^2$ in ΔY_t , which is another source of randomness in the expansion. And this gives us another terms to the "Ito formula". In order to present and prove the lemma, we shall discretize the processes $\{W_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ by restricting t to \mathcal{D} . And let us write for all $t \in \mathcal{D}$.

$$\Delta W_t = W_{t+\Delta t} - W_t, \quad \text{and} \quad \Delta X_t = X_{t+\Delta t} - X_t.$$

The next lemma is a version of Rao et al. (1974).

Lemma 1 If $\rho(t, x)$ is continuously differentiable, then

$$\Delta X_t = b(t, X_t)\Delta t + \rho(t, X_t)\Delta W_t + \rho_{01}(t, X_t)^2(\Delta W_t)^2 + o_p(\Delta t) \text{ as } \Delta t \rightarrow 0 \quad (2.4)$$

where $\rho_{01}(t, x) = \frac{\partial}{\partial x} \rho(t, x)$. Moreover if $g(t, x)$ is three times continuously differentiable in (t, x) , then

$$\begin{aligned} \Delta Y_t = & \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2 g_{02}(t, X_t) \right] \Delta t \\ & + \left[\rho(t, X_t)g_{01}(t, X_t) \right] \Delta W_t \\ & + \left[\left\{ \frac{1}{2}\rho(t, X_t)^2 g_{02}(t, X_t) + \rho_{01}(t, X_t)^2 g_{01}(t, X_t) \right\} \Delta t \right] \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right] \\ & + \text{Rem}_1, \end{aligned} \quad (2.5)$$

where Rem_1 consists of terms of the order higher than or equal to $(\Delta t)^{\frac{3}{2}}$ and $(\Delta W_t)^3$. We note that Rem_1 converges to zero in probability with the order of magnitude $O_p(\Delta t)^{\frac{3}{2}}$ as $\Delta t \rightarrow 0$.

Proof. The proof of (2.4) is straightforward and it is similar to (2.5). We shall give only the proof of (2.5). If (2.4) holds, then by expanding $g(t + \Delta t, X_{t+\Delta t})$ about (t, X_t) , we obtain

$$\begin{aligned} g(t + \Delta t, X_{t+\Delta t}) &= g(t, X_t) + g_{10}(t, X_t)\Delta t + g_{01}(t, X_t)\Delta X_t \\ &\quad + \frac{1}{2} \{g_{20}(t, X_t)(\Delta t)^2 + 2g_{11}(t, X_t)(\Delta t)(\Delta X_t) \\ &\quad + g_{02}(t, X_t)(\Delta X_t)^2\} + \text{Rem}_2, \end{aligned} \quad (2.6)$$

where Rem_2 consists of terms of order at most as big as $O_p\{(\Delta t)^i(\Delta X_t)^j\}$, $i, j \geq 0$, $i + j = 3$ as $\Delta t \rightarrow 0$. It follows that

$$\begin{aligned} \Delta Y_t &= \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2g_{02}(t, X_t) \right] \Delta t \\ &\quad + \rho(t, X_t)g_{01}(t, X_t)\Delta W_t \\ &\quad + \left[\frac{1}{2}\rho(t, X_t)^2g_{02}(t, X_t) + \rho_{01}(t, X_t)^2g_{01}(t, X_t) \right] \Delta t \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right] \\ &\quad + \text{Rem}_1^*, \text{ (say),} \end{aligned} \quad (2.7)$$

where Rem_1^* is of the term as big as $O_p\{(\Delta t)(\Delta W_t)\} = O_p\{(\Delta t)^{\frac{3}{2}}\}$. By the weak law of large numbers, it is easily seen that $[(\Delta t)^2 - \Delta t] = O_p(\Delta t)$ as $\Delta t \rightarrow 0$. Rewriting it as $(\Delta t) \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right]$, we have shown the lemma. ///

It is easily seen that random parts, ΔW_t and $\left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right]$, in (2.5) are uncorrelated. Moreover, as we shall see below (cf (2.9)), $\sqrt{\Delta t} \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right]$ behaves as if this is another Brownian motion process. These facts suggests us to write $(\Delta t) \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right]$ as $\sqrt{h}\sqrt{\Delta t} \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right]$, so that (2.5) may be rewritten as

$$\begin{aligned} \Delta Y_t &= \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2g_{02}(t, X_t) \right] \Delta t \\ &\quad + [\rho(t, X_t)g_{01}(t, X_t)] \Delta W_t \\ &\quad + \left[\frac{1}{2}\rho(t, X_t)^2g_{02}(t, X_t) + \rho_{01}(t, X_t)^2g_{01}(t, X_t) \right] \sqrt{h}\sqrt{\Delta t} \left[\frac{(\Delta W_t)^2}{\Delta t} - 1 \right] \\ &\quad + \text{Rem}_1 \end{aligned} \quad (2.8)$$

which may be more convenient for our purpose.

In order to make the methods of the continuous time models applicable to a discrete model, we shall consider a continuous time approximation to the formula (2.5) and (2.8). Our

method of approximation is based on the following heuristic argument which may be justified mathematically elsewhere. Although, in obtaining the Ito formula, we should interpret dW_t as a L_2 limit of ΔW_t in the stochastic integral, we shall write $\Delta W_t = \sqrt{\Delta t}Z$, where Z is the standard normal random variable. In view of this interpretation, the second random term in (2.5), $[\Delta W_t^2 - \Delta t]$, may be written as $(\Delta t)[Z^2 - 1]$. We shall write $\Delta t = \sqrt{h}\sqrt{\Delta t}$ as in (2.8) and we shall fix h throughout. The reasons for this fuzzy trick are that we shall retain the smallest time interval between any consecutive trades on the one hand, and we shall obtain the diffusion approximation to $\sqrt{\Delta t}\left[\frac{\Delta W_t^2}{\Delta t} - 1\right]$ as $\Delta t \rightarrow 0$ on the other hand. Since the joint distribution of $\left(\Delta W_t, \sqrt{h}\sqrt{\Delta t}\left[\frac{\Delta W_t^2}{\Delta t} - 1\right]\right)$ and $(\sqrt{\Delta t}Z, \sqrt{h}\sqrt{\Delta t}[Z^2 - 1])$ are the same, the joint moment generating function $g(\theta_1, \theta_2)$ of $\left(\Delta W_t, \sqrt{h}\sqrt{\Delta t}\left[\frac{\Delta W_t^2}{\Delta t} - 1\right]\right)$ becomes

$$\begin{aligned} g(\theta_1, \theta_2) &= E \left\{ \exp \left[\theta_1 \Delta W_t + \theta_2 \sqrt{h}\sqrt{\Delta t} \left[\frac{\Delta W_t^2}{\Delta t} - 1 \right] \right] \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left\{ \theta_1 \sqrt{\Delta t} x + \theta_2 \sqrt{h}\sqrt{\Delta t} (x^2 - 1) \right\} \exp \left\{ -\frac{1}{2} x^2 \right\} dx \\ &= \sqrt{\frac{1}{1 - 2\theta_2 \sqrt{h}\Delta t}} \exp \left\{ -\theta_2 \sqrt{h}\Delta t + \frac{1}{2} \frac{\theta_1^2 \Delta t}{1 - 2\theta_2 \sqrt{h}\Delta t} \right\} \\ &= \exp \left\{ \frac{1}{2} (\Delta t) \theta_1^2 + \frac{1}{2} (2\Delta t) h \theta_2 + O \left((\Delta t)^{\frac{3}{2}} \right) \right\} \quad \text{as } \Delta t \rightarrow 0. \end{aligned} \quad (2.9)$$

It follows that the joint distribution of $\left(\Delta W_t, \sqrt{h}\sqrt{\Delta t}\left[\frac{\Delta W_t^2}{\Delta t} - 1\right]\right)$ may be approximated by the bivariate normal distribution with the mean vector $(0, 0)'$ and the variance covariance matrix

$$\begin{pmatrix} \Delta t, & 0 \\ 0, & 2(\Delta t)h \end{pmatrix}, \quad (2.10)$$

up to the order of magnitude as big as (Δt) as $\Delta t \rightarrow 0$. We have thus obtained,

Lemma 2. Let $\{Z(t), t \geq 0\}$ be a standard Brownian motion process independent of $\{W(t), t \geq 0\}$ then,

$$\begin{aligned} \Delta Y_t &= \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2 g_{02}(t, X_t) \right] \Delta t \\ &\quad + \rho(t, X_t)g_{01}(t, X_t)\Delta W_t \\ &\quad + \frac{\sqrt{h}}{\sqrt{2}} \left[\rho(t, X_t)^2 g_{02}(t, X_t) \sqrt{2}\rho_{01}(t, X_t)g_{01}(t, X_t) \right] \Delta Z_t + O_p(\Delta t)^{\frac{3}{2}} \end{aligned} \quad (2.11)$$

as $\Delta t \rightarrow 0$, where $Z_t = Z(t)$, $\Delta Z_t = Z(t + \Delta t) - Z(t)$.

Note: If we let $h \rightarrow 0$ and $\Delta t \rightarrow 0$ in (2.11), then, via Ito integral, we may formally eliminate the terms involving $\sqrt{h}\Delta Z_t$. The term, however, is the leading term of the random fluctuation which results solely from the discrete time model. In order to retain the random

fluctuation, but at the same time, to enjoy the power of stochastic calculus, we shall let $\Delta t \rightarrow 0$, while h is held fixed. This mathematically rather fuzzy operation may lead us to the following modification of the Ito formula (the discrete Ito formula).

$$\begin{aligned} (dY_t)^* = & \left[g_{10}(t, X_t) + b(t, X_t)g_{01}(t, X_t) + \frac{1}{2}\rho(t, X_t)^2 g_{02}(t, X_t) \right] dt \\ & + \rho(t, X_t)g_{01}(t, X_t)dW_t \\ & + \frac{\sqrt{h}}{\sqrt{2}} \left[\rho(t, X_t)^2 g_{02}(t, X_t) + \sqrt{2}\rho_{01}(t, X_t)^2 g_{01}(t, X_t) \right] dZ_t. \end{aligned} \quad (2.12)$$

III. The Discrete Time One Factor Model

We shall consider a discrete time and continuous state version of Vasicek's one factor model (Vasicek (1976)). For this purpose, we shall first formulate the problem in the continuous time domain. We then discretize the model, so that the model is consistent with the observation.

Let, (Ω, \mathcal{F}, Q) be a probability space and let $[0, \tau]$ be an interval of time, where τ denotes the fixed finite horizon. Let $P^T(t)$ be a time t price of default free T maturity zero coupon bond, and $r(t)$ be an instantaneous riskless spot interest rate at time t . These processes may be continuous time and continuous state stochastic processes. We shall, however, suppose that $P^T(t)$ and $r(t)$ may be observed only at the time of trading. As in Section 1, we suppose that there is a positive constant $h > 0$, which is the shortest time span between any consecutive trades. We also let $\mathcal{D} = \{t_0, t_1, \dots, T^*\}$, $t_0 = 0$, $t_n = T^*$, $t_{i+1} - t_i = \Delta t_i$ ($i = 0, 1, \dots, n$) be the set of all time points at which the price of all securities may be observed, and $T^* \in (\tau - h, \tau]$ is the final trading day of our economy. And without loss of generality we shall assume $T^* = \tau$ in the rest of the paper. We shall set for simplicity that $\Delta t_i = h$ for all $0 \leq i \leq n$ to obtain the basic equation, we then separate the roll of h and Δt_i 's later in this section. In the continuous time domain, $P^T(t)$ and $r(t)$ are assumed to be continuous stochastic processes satisfying the following

Assumption 1. (i) $r_t = r(t)$ is a continuous Markov process defined by Ito integral,

$$dr(t) = b(t, r_t)dt + \rho(t, r_t)dW(t), \quad W(t) \sim BM(0, 1). \quad (3.1)$$

(ii) $P^T(t)$ is a stochastic process determined by $r(t)$,

$$P^T(t) = P^T(t, r_t). \quad (3.2)$$

(iii) There are no arbitrage opportunities in the market.

Assumption 1 is the same as Vasicek (1977)'s. In order to discretize the model, we need

Assumption 2. (i) The values of $r(t)$ and $P^T(t)$ are observed only at these t and T in \mathcal{D} .

(ii) As a function of $(t, r) \in [0, \tau] \times R$, $P^T(t, r)$ is three times continuously differentiable in (t, r) for all T .

From Assumption 1-(iii) and Assumption 2-(i), we need to consider the local no arbitrage condition on the set \mathcal{D} . And for this purpose, as in (2.4) we shall restate the discrete version of (3.1),

$$\begin{aligned}\Delta r(t) &= r(t + \Delta t) - r(t) \\ &= b(t, r_t)\Delta t + \rho(t, r_t)\Delta W_t + \rho_{01}(t, r_t)^2(\Delta W_t)^2 + O_p(\Delta t), \quad t \in \mathcal{D},\end{aligned}\quad (3.3)$$

Note that $\Delta W(t)$ is normally distributed with mean zero and variance Δt . It follows from Lemma 1 that for all $t, T \in \mathcal{D}$, $\Delta P^T(t, r_t) = P^T(t + \Delta t, r_{t+\Delta t}) - P^T(t, r_t)$ becomes

$$\begin{aligned}\Delta P^T(t, r_t) &= \left[P_{10}^T(t, r_t) + b(t, r_t)P_{01}^T(t, r_t) + \frac{1}{2}\rho(t, r_t)^2 P_{02}^T(t, r_t) \right] \Delta t \\ &\quad + \rho(t, r_t)P_{01}^T(t, r_t)\Delta W_t + \left[\frac{1}{2}\rho(t, r_t)^2 P_{02}^T(t, r_t) + \rho_{01}(t, r_t)^2 P_{01}^T(t, r_t) \right] \\ &\quad \times \sqrt{h} \left[\sqrt{\Delta t} \left(\frac{(\Delta W(t))^2}{\Delta t} - 1 \right) \right] + \text{Rem}_1,\end{aligned}\quad (3.4)$$

where Rem_1 consists of terms with the order of magnitude as big as $\{(\Delta t)^{\frac{3}{2}} + (\Delta W_t)^3\}$ as $\Delta t \rightarrow 0$, and $P_{ij}^T(t) = P_{ij}^T(t, r) = \frac{\partial^{i+j}}{\partial t^i \partial r^j} P^T(t, r)$, $i + j \leq 2$, $i, j \geq 0$. Disregarding the remainder term, (3.4) becomes

$$\Delta P^T(t) = \mu^T(t)P^T(t)\Delta t + \sigma_1^T(t)P^T(t)\Delta W(t) + \sigma_2^T(t)P^T(t)\Delta Z(t), \quad (3.5)$$

where,

$$\begin{aligned}\Delta Z(t) &= \sqrt{\Delta t} \left\{ \frac{(\Delta W(t))^2}{\Delta t} - 1 \right\} \\ \mu^T(t) &= \frac{1}{P^T(t)} \left\{ P_{10}^T(t) + b(t, r_t)P_{01}^T(t) + \frac{1}{2}\rho(t, r_t)^2 P_{02}^T(t) \right\} \\ \sigma_1^T(t) &= \frac{1}{P^T(t)} \rho(t, r_t)P_{01}^T(t)\end{aligned}\quad (3.6)$$

and

$$\sigma_2^T(t) = \frac{\sqrt{h}}{P^T(t)} \left\{ \frac{1}{2}\rho(t, r_t)^2 P_{02}^T(t) + \rho_{01}(t, r_t)^2 P_{01}^T(t) \right\}.$$

It is interesting to note that the increment $\Delta P^T(t) = \Delta P^T(t, r_t)$ has two uncorrelated random parts. The first part, $\sigma_1^T(t)P^T(t)\Delta W(t)$, is the discrete version of the usual random part in the continuous time models. The second part is $\sigma_2^T(t)P^T(t)\Delta Z(t)$, which is of the order of magnitude \sqrt{h} , and this will be considered as the correction term. And this term should be included to obtain the basic local no arbitrage conditions. This suggests us that the discrete one factor model may be analyzed as if it is a two factor continuous model. And we shall discuss it in the line of Brennan and Schwartz (1979), Miura and Kishino (1995) and Takahashi (1996).

Let us consider a portfolio which invests an amount $V_1(t)$ of a bond with maturity date T_1 , $V_2(t)$ with maturity date T_2 and $V_3(t)$ with maturity date T_3 at time t . Then the total value of the portfolio is

$$V(t) = V_1(t) + V_2(t) + V_3(t).$$

We shall write the number of T_i -maturity bond in the portfolio at time t by

$$\omega_i(t) = \frac{V_i(t)}{P^{T_i}(t)}, \quad i = 1, 2, 3. \quad (3.7)$$

It follows that,

$$V(t) = \omega_1(t)P^{T_1}(t) + \omega_2(t)P^{T_2}(t) + \omega_3(t)P^{T_3}(t). \quad (3.8)$$

The values of $\omega_i(t)$'s are determined at time t , and they will be held fixed until the values $P^{T_i}(t + \Delta t)$ are observed. Hence from (3.4) ~ (3.6) and (3.8), we have, for all $t \in \mathcal{D}$,

$$\begin{aligned} \Delta V(t) &= I(t)\Delta t + II_1(t)\Delta W(t) + II_2(t)\Delta Z(t) + \text{Rem}_1^* \\ &= \Delta V(t)^* + \text{Rem}_1^* \quad (\text{say}), \end{aligned} \quad (3.9)$$

where $\Delta V(t) = V(t + \Delta t) - V(t)$, and Rem_1^* is the term of the order as big as $O_p\left\{(\Delta t)^{\frac{3}{2}} + (\Delta W)^3\right\}$. Here we have set,

$$\begin{aligned} I(t) &= \{\omega_1(t)\mu^{T_1}(t)P^{T_1}(t) + \omega_2(t)\mu^{T_2}(t)P^{T_2}(t) + \omega_3(t)\mu^{T_3}(t)P^{T_3}(t)\} \\ II_1(t) &= \{\omega_1(t)\sigma_1^{T_1}(t)P^{T_1}(t) + \omega_2(t)\sigma_1^{T_2}(t)P^{T_2}(t) + \omega_3(t)\sigma_1^{T_3}(t)P^{T_3}(t)\} \end{aligned}$$

and

$$II_2(t) = \{\omega_1(t)\sigma_2^{T_1}(t)P^{T_1}(t) + \omega_2(t)\sigma_2^{T_2}(t)P^{T_2}(t) + \omega_3(t)\sigma_2^{T_3}(t)P^{T_3}(t)\}.$$

Next, we shall choose ω_i 's so that $II_1(t) = II_2(t) = 0$. Then the resulting portfolio will be risk free between t and $t + \Delta t$ up to the order $O_p\{(\Delta t)^{\frac{3}{2}}\}$. It follows that

$$\omega_1(t) = \left(\frac{D_3}{D_2}\right) \left(\frac{P^{T_3}(t)}{P^{T_1}(t)}\right) \omega_3(t)$$

and

$$\omega_2(t) = \left(\frac{D_1}{D_2}\right) \left(\frac{P^{T_3}(t)}{P^{T_2}(t)}\right) \omega_3(t),$$

where

$$\begin{aligned} D_1 &= \sigma_1^{T_1}(t)\sigma_2^{T_3}(t) - \sigma_1^{T_3}(t)\sigma_2^{T_1}(t), \\ D_2 &= \sigma_1^{T_2}(t)\sigma_2^{T_1}(t) - \sigma_1^{T_1}(t)\sigma_2^{T_2}(t), \end{aligned}$$

and

$$D_3 = \sigma_1^{T_3}(t)\sigma_2^{T_2}(t) - \sigma_1^{T_2}(t)\sigma_2^{T_3}(t).$$

If $V(t) \geq 0$, then the local no arbitrage condition becomes

$$\Delta V(t) = V(t)r(t)\Delta t. \quad (3.10)$$

It follows that there is no arbitrage opportunity in the market if and only if the equation

$$D_3 [\mu^{T_1}(t) - r(t)] + D_1 [\mu^{T_2}(t) - r(t)] + D_2 [\mu^{T_3}(t) - r(t)] = 0, \quad (3.11)$$

holds for any T_1, T_2 , and T_3 in \mathcal{D} . By the standard results in linear algebra, it can be shown that (3.11) holds if there exist functions $\lambda_1(t)$ and $\lambda_2(t)$, for which,

$$\begin{aligned} \sigma_1^T(t)\lambda_1(t) + \sigma_2^T(t)\lambda_2(t) &= \mu^T(t) - r(t) \\ \sigma_1^S(t)\lambda_1(t) + \sigma_2^S(t)\lambda_2(t) &= \mu^S(t) - r(t) \end{aligned} \quad (3.12)$$

for any T and S in \mathcal{D} . Note that functions λ_1 and λ_2 do not depend on T and S . The functions $(-\lambda_1(t), \lambda_2(t))$ may be called the market prices of risk in the discrete model. From the definitions of $\sigma_1^T(t)$, $\sigma_2^T(t)$, and $\mu^T(t)$, we have obtained the basic equation for $P^T(t, r_t)$,

$$\begin{aligned} P_{10}^T(t, r) + \left[b(t, r_t) - \rho(t, r_t)\lambda_1(t) - \sqrt{h}\rho_{01}(t, r_t)^2\lambda_2(t) \right] P_{01}^T(t, r_t) \\ + \frac{1}{2}\rho(t, r_t)^2 \left[1 - \sqrt{h}\lambda_2(t) \right] P_{02}^T(t, r_t) - r(t)P^T(t, r_t) = 0, \end{aligned} \quad (3.13)$$

with boundary condition

$$P^T(T, r_T) = 1. \quad (3.14)$$

This is valid up to the order of magnitude $O_p(h)$: Here, if we shall let $\Delta t \rightarrow 0$ while h being held fixed, then via (2.12) we may formally interpret (3.13) as a partial differential equation for $P^T(t, r_t)$. And we shall denote its solution by $\tilde{P}^T(t, r_t)$ below. We may also assume that the value of h may be relatively smaller, so that $(1 - \sqrt{h}\lambda_2(t)) > 0$ may hold. The solutions of (3.13) and (3.14) may be obtained numerically.

By the Feynman-Kac formula, a solution to the "PDE" (3.13) with the boundary condition (3.14) is given by

$$\tilde{P}^T(t, r_t) = E^Q \left\{ \exp \left\{ - \int_t^T X(u) du \right\} \middle| \mathcal{F}_t \right\}, \quad (3.15)$$

where $X(u)$ is the Ito diffusion defined by

$$\begin{aligned} dX(u) &= \left[b(u, r_u) - \rho(u, r_u)\lambda_1(u) - \sqrt{h}\rho_{01}(u, r_u)^2\lambda_2(u) \right] du \\ &\quad + \rho(u, r_u)\sqrt{1 - \sqrt{h}\lambda_2(u)}dW(u). \end{aligned} \quad (3.16)$$

If $\lambda_1(t) = \lambda_2(t) = 0$, then $\tilde{P}^T(t, r_t)$ agrees with the Vasicek's solution in the risk neutral world.

IV. A Special Case

It is not easy to implement the result of the previous section in its general form. We shall consider the simplest case where the market price of risks $(-\lambda_1(t), \lambda_2(t))$ and the volatility parameter ρ are constant. Let

$$\lambda_1(t) = \lambda_1, \quad \lambda_2(t) = \lambda_2, \quad (4.1)$$

be independent of t and $r(t)$. We shall also assume that $r(t)$ follows the Ornstein-Uhlenbeck process,

$$dr(t) = \alpha(\gamma - r(t))dt + \rho dW(t), \quad (4.2)$$

where α and ρ are positive constants. In the case of non-stochastic volatility like this example, (3.13) may be substantially simplified. For example $P^T(t)$ is the solution of,

$$P_{10}^T(t, r_t) + [\alpha(\gamma - r(t))\lambda_1]P_{01}^T(t, r_t) - \frac{1}{2}\rho^2[1 - \sqrt{h}\lambda_2]P_{02}^T(t, r_t) - r(t)P_{10}^T = 0. \quad (4.3)$$

This may be solved at least numerically.

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